Calculus II - Day 22

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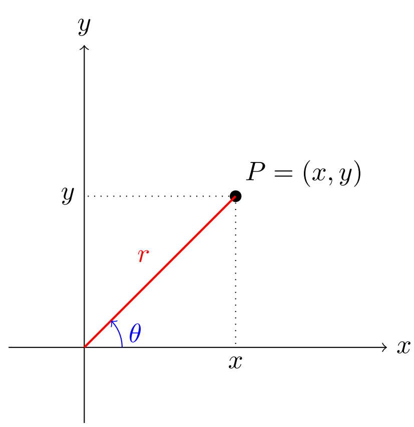
Goals for Today

- Plot points and curves in the polar coordinate system.
- Convert between polar and Cartesian (rectangular) coordinates.

Cartesian Coordinates

A point in Cartesian coordinates is represented as $P = (x, y)$.

Polar Coordinates



Another way to represent the point: polar coordinates (r, θ).

- $r =$ distance from the origin.
- $\theta =$ angle between the positive x-axis and the line segment through the point and the origin.

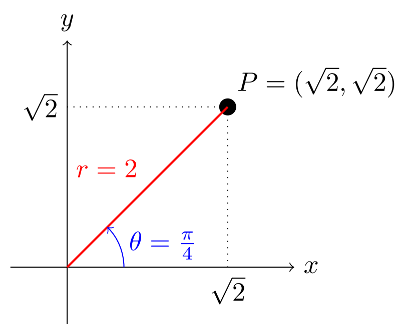
Example

Represent the point $(x, y) = (\sqrt{2}, \sqrt{2})$ in polar coordinates:

$$r = \sqrt{x^2 + y^2} = \sqrt{\sqrt{2}^2 + \sqrt{2}^2} = \sqrt{4} = 2$$

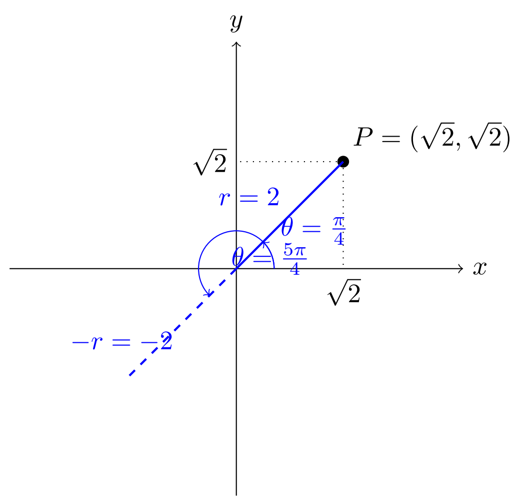
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\pi}{4}$$

The polar coordinates are $(r, \theta) = (2, \frac{\pi}{4})$.



Alternative Representation

Yet, this point can also be represented as $(-2, \frac{5\pi}{4})$.



And, this can also be represented as $(r, \theta) = (2, \frac{9\pi}{4}) = (2, -\frac{7\pi}{4})$, since adding or subtracting 2π from θ corresponds to a full rotation, bringing the point back to the same position.

Finding Other Representations of $P = (r, \theta)$

- Add or subtract an integer multiple of 2π from θ:

$$(r, \theta + 2n\pi)$$

- Add or subtract an integer multiple of π from θ and negate r:

$$(-r, \theta + (2n+1)\pi)$$

While Cartesian representations are unique, <u>polar coordinates are not</u>.

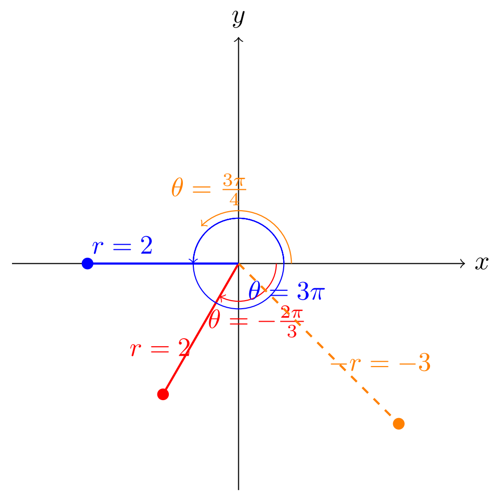
Example: Plot Points in Polar Coordinates

Plot the following points with polar coordinates:

(a) $(2, -\frac{2\pi}{3})$

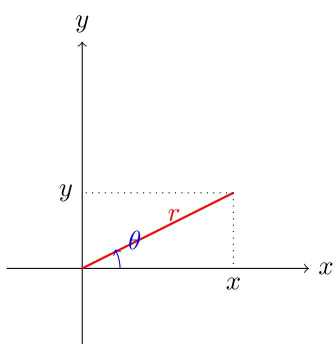
(b) $(2, 3\pi)$

(c) $(-3, \frac{3\pi}{4})$



Converting from Polar to Cartesian Coordinates

To convert polar coordinates (r, θ) to Cartesian coordinates (x, y):



$$\cos(\theta) = \frac{x}{r} \quad \rightarrow \quad x = r\cos(\theta)$$

$$\sin(\theta) = \frac{y}{r} \quad \rightarrow \quad y = r\sin(\theta)$$

Example: Converting Polar to Cartesian Coordinates

Let $(r, \theta) = (2, \frac{\pi}{3})$. Convert to Cartesian coordinates:

$$x = r\cos(\theta) = 2\cos\left(\frac{\pi}{3}\right) = 2 \cdot \frac{1}{2} = 1$$

$$y = r\sin(\theta) = 2\sin\left(\frac{\pi}{3}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

So, the Cartesian coordinates are:

$$(x, y) = (1, \sqrt{3})$$

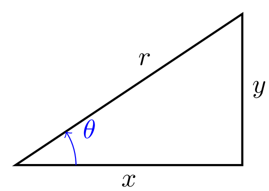
Converting Cartesian to Polar Coordinates

Given (x, y):

$$r = \sqrt{x^2 + y^2}$$

$$\tan(\theta) = \frac{y}{x}$$

There are two angles in $[0, 2\pi)$ or $(-\pi, \pi]$, etc., with the same tangent. Which one is correct? The correct quadrant of θ depends on the signs of x and y.



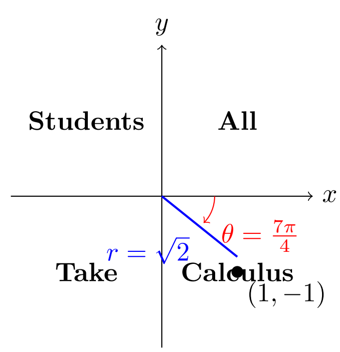
Example: Converting Cartesian to Polar Coordinates

Convert $(x, y) = (1, -1)$ to polar coordinates:

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan(\theta) = \frac{y}{x} = \frac{-1}{1} = -1$$

There are two angles θ with $\tan(\theta) = -1$: $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$. The correct angle depends on the quadrant of (x, y). Since $x > 0$ and $y < 0$, the point is in the fourth quadrant, so $\theta = \frac{7\pi}{4}$.



Representing Curves in Polar Coordinates

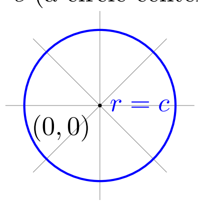
In polar coordinates, instead of $y = f(x)$, we usually use $r = f(\theta)$ to represent polar curves.

- θ: Independent variable.
- r: Dependent variable.

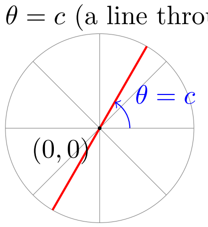
The equation $r = f(\theta)$ describes "how the distance depends on the angle."

Basic Graph Examples

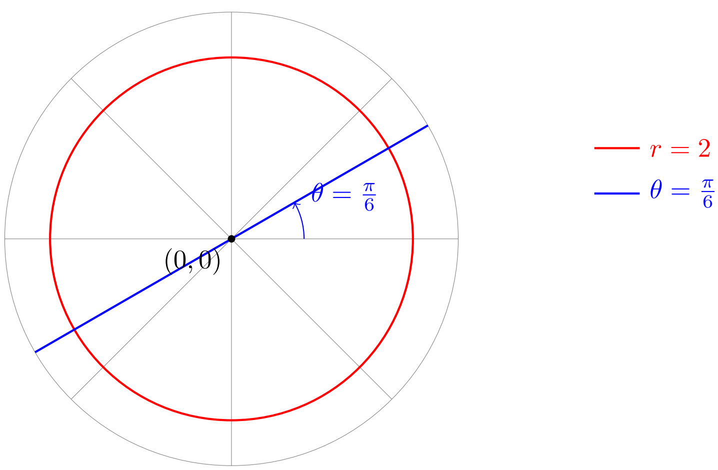
$r = c$ (a circle centered at $(0, 0)$ with radius c).



$\theta = c$ (a line through the origin making an angle θ with the positive x-axis).



Example: Representing Polar Curves in Cartesian Coordinates



Converting $r = 2$ to Cartesian:

$$x^2 + y^2 = r^2 \quad \text{but } r = 2$$

$$x^2 + y^2 = 4 \quad \boxed{x^2 + y^2 = 4}$$

Converting $\theta = \frac{\pi}{6}$ to Cartesian:

$$\frac{y}{x} = \tan(\theta) \quad \text{but } \theta = \frac{\pi}{6}$$

$$\frac{y}{x} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \quad \to \quad y = \frac{1}{\sqrt{3}}x$$

More Basic Graphs in Polar Coordinates

Horizontal Line: $y = c$

$$y = r\sin(\theta) \quad \to \quad r\sin(\theta) = c$$

$$r = \frac{c}{\sin(\theta)} = c\csc(\theta)$$

Vertical Line: $x = c$

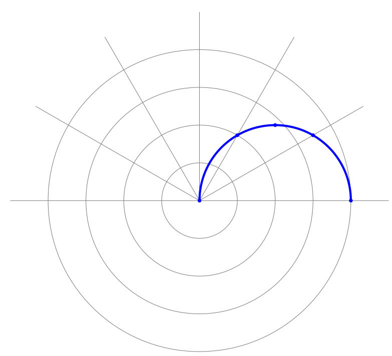
$$x = r\cos(\theta) \quad \to \quad r\cos(\theta) = c$$

$$r = \frac{c}{\cos(\theta)} = c\sec(\theta)$$

Example: Sketching the Polar Function $r = 2\cos(\theta)$

Table of Values:

θ	$r = 2\cos(\theta)$
0	$2 \cdot \cos(0) = 2$
$\frac{\pi}{6}$	$2 \cdot \cos\left(\frac{\pi}{6}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$
$\frac{\pi}{4}$	$2 \cdot \cos\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$
$\frac{\pi}{3}$	$2 \cdot \cos\left(\frac{\pi}{3}\right) = 2 \cdot \frac{1}{2} = 1$
$\frac{\pi}{2}$	$2 \cdot \cos\left(\frac{\pi}{2}\right) = 2 \cdot 0 = 0$



Now, what happens if we try more values for θ? From the graph, we can infer that the polar curve $r = 2\cos(\theta)$ creates a circle offset from the x-axis.

Let's convert this to rectangular.

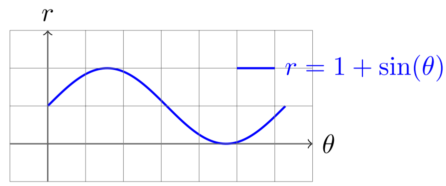
$$r = 2\cos(\theta) \quad \to \quad \cos(\theta) = \frac{r}{2}$$

$$r\cos(\theta) = \frac{r^2}{2} \quad \to \quad x = \frac{r^2}{2} \quad \to \quad r^2 = 2x$$

$$x^2 + y^2 = r^2 \quad \to \quad x^2 + y^2 = 2x$$

Example: Graphing the Polar Function $r = 1 + \sin(\theta)$ (Cardioid)

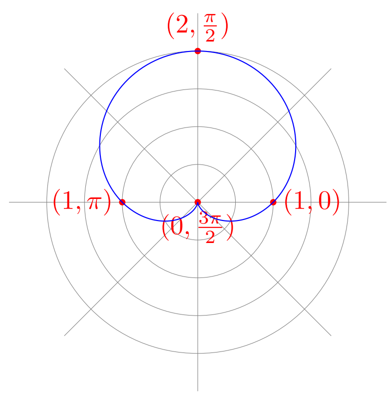
Sometimes it is useful to treat r like y and θ like x to interpret the distances from the origin more easily.



Plotting Important Points from $r = 1 + \sin(\theta)$

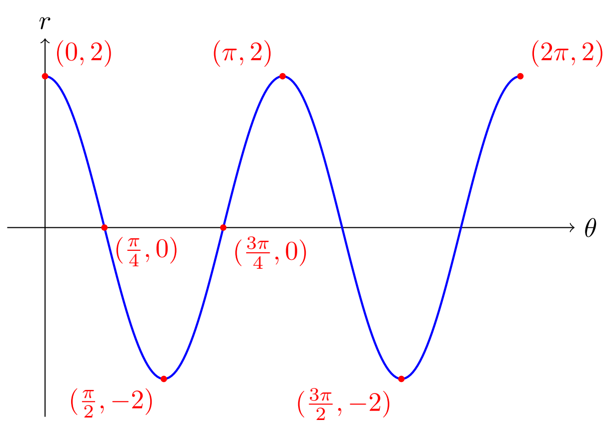
From the rectangular graph, we identify key points:

$$\text{Polar Coordinates: } (r, \theta) = (1, 0), (2, \frac{\pi}{2}), (1, \pi), (0, \frac{3\pi}{2})$$



Example: The 4-Petaled Rose $r = 2\cos(2\theta)$

Rectangular Plot:



Polar Plot: